Efficient evaluation of highly oscillatory acoustic scattering surface integrals

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Abstract

We consider the approximation of highly oscillatory weakly singular surface integrals, arising from boundary integral methods for solving high frequency acoustic scattering problems in three dimensions. As the frequency of the incident wave increases, the performance of standard quadrature schemes deteriorates. Naive application of asymptotic schemes also fails due to the weak singularity. We present here a new scheme based on a combination of an asymptotic approach and exact treatment of singularities in an appropriate coordinate system. We demonstrate that the computational cost of evaluating many integrals over the surface of a scatterer does not increase significantly as the frequency increases.

Key words: high frequency scattering, acoustics, oscillatory integrals, method of stationary phase
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1 Introduction

This paper is concerned with the approximation of integrals of the form

$$M\psi(\mathbf{x}) := \int_{\partial D} \frac{m(\mathbf{x},\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} \mathrm{e}^{\mathrm{i}k[|\mathbf{x}-\mathbf{y}|+\hat{\mathbf{d}}.(\mathbf{y}-\mathbf{x})]}\psi(\mathbf{y})\,\mathrm{d}s(\mathbf{y}), \qquad \mathbf{x}\in\partial D, \tag{1}$$

where $m(\mathbf{x},\mathbf{y})$, $\psi(\mathbf{y})$ are smooth and slowly oscillating functions, $\hat{\mathbf{d}}$ is a fixed unit vector (the incident wave direction), and ∂D is the surface of a three dimensional convex obstacle D. Such integrals arise in boundary integral methods for acoustic scattering problems, and if the acoustic size kA is large (where A is the size of obstacle), corresponding to the high frequency problem, the integrand will be highly oscillatory. For simulation of scattered acoustic waves, evaluation of (1) is required for many observation directions $\mathbf{x}$, and when kA is large the cost of doing this by standard quadrature schemes is prohibitive.

Much recent research has focused on developing efficient quadratures for highly oscillatory integrals (see for example [5] and the references therein). Although many excellent schemes have been developed, their application to weakly singular surface integrals of the form (1) is still being investigated.

Here, instead of evaluating (1) using quadrature, we consider ideas based on a rotated coordinate system and a localized *Method of Stationary Phase (MSP)*. In general, a direct MSP evaluation of (1) breaks down because of the weak singularity. Our rotated coordinate system approach avoids this difficulty and leads to an asymptotic expansion of (1) in ascending powers of $1/k$. Approximating (1) by the first few terms in the expansion we can determine a bound on the error converging to zero as k tends to infinity.

For a given wavenumber k and observed direction $\mathbf{x}\in\partial D$, the main aim of this paper is to devise a formula to compute (1) within a few seconds, with the CPU time not increasing as the frequency increases. First we describe the acoustic scattering problem leading to (1).

Consider scattering of a time-harmonic acoustic plane wave u^i by a sound soft bounded convex obstacle $D\subset\mathbb{R}^3$ with smooth surface ∂D described in spherical coordinates. We seek an approximation to the *radiating* solution u of the exterior Helmholtz problem

$$\Delta u + k^2 u = 0, \quad \text{in } \mathbb{R}^3\backslash\overline{D}, \qquad u = -u^i := -\mathrm{e}^{\mathrm{i}k\mathbf{x}.\hat{\mathbf{d}}}, \quad \text{on } \partial D. \tag{2}$$

Although problems such as this have a long pedigree, there is considerable interest in establishing reliable numerical schemes for the high frequency case.

In particular, for scattering by convex three dimensional obstacles with smooth boundaries a number of very efficient high order boundary integral schemes have recently been proposed [2,4]. These schemes exhibit extremely fast (superalgebraic or even exponential) convergence rates for frequencies starting from the resonance region to a medium level (size of the obstacle is about a hundred times the wavelength). But they break down for shorter wavelengths. One of the main reasons for this is the expense of evaluating many highly oscillatory integrals of the form (1) in the scheme.

The unique radiating solution u of (2) can be represented as [3, p.59]

$$u(\mathbf{x}) = -\int_{\partial D} \Phi(\mathbf{x},\mathbf{y})v(\mathbf{y})\,\mathrm{d}s(y), \quad \mathbf{x} \in \mathbb{R}^3\backslash\overline{D},$$

where $\Phi(\mathbf{x},\mathbf{y}) := \mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{y}|}/(4\pi|\mathbf{x}-\mathbf{y}|)$, and the unknown density $v \in C(\partial D)$ is the unique solution of the boundary integral equation

$$\frac{1}{2}v(\mathbf{x}) + \int_{\partial D}\left[\frac{\partial\Phi(\mathbf{x},\mathbf{y})}{\partial\mathbf{n}(\mathbf{x})} - \mathrm{i}\Phi(\mathbf{x},\mathbf{y})\right]v(\mathbf{y})\,\mathrm{d}s(\mathbf{y}) = \frac{\partial u^i}{\partial\mathbf{n}}(\mathbf{x}) - \mathrm{i}u^i(\mathbf{x}),\ \mathbf{x}\in\partial D. \quad (3)$$

We write $v(\mathbf{x}) = \phi(\mathbf{x})\mathrm{e}^{\mathrm{i}k\mathbf{x}.\hat{d}}$, where ϕ is slowly oscillating compared to $\mathrm{e}^{\mathrm{i}k\mathbf{x}.\hat{d}}$ (see e.g. [1]). This reduces (3) to the second kind boundary integral equation

$$\phi(\mathbf{x}) + \int_{\partial D}\frac{m(\mathbf{x},\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\mathrm{e}^{\mathrm{i}k[|\mathbf{x}-\mathbf{y}|+\hat{\mathbf{d}}.(\mathbf{y}-\mathbf{x})]}\phi(\mathbf{y})\,\mathrm{d}s(\mathbf{y}) = 2\mathrm{i}(k\mathbf{n}(\mathbf{x})\cdot\hat{\mathbf{d}} - 1), \quad (4)$$

where $\mathbf{n}(\mathbf{x})$ is the unit outward normal vector to the surface ∂D at $\mathbf{x}$ and $m(\mathbf{x},\mathbf{y})$ is a smooth function, given by

$$m(\mathbf{x},\mathbf{y}) := \frac{1}{4\pi}\left[\frac{\mathbf{n}(\mathbf{x})(\mathbf{y}-\mathbf{x})^T}{|\mathbf{x}-\mathbf{y}|^2}(1-\mathrm{i}k|\mathbf{x}-\mathbf{y}|) - \mathrm{i}\right]. \quad (5)$$

In any numerical scheme to solve (4), with unknown function $\phi(\mathbf{x})$ approximated in a finite dimensional space by $\phi_L(\mathbf{x}) := \sum_{j=1}^{L} v_j\rho_j(\mathbf{x})$, we are faced with the difficulty of evaluation of highly oscillatory integrals of the form (1) with density ψ replaced by the basis function ρ_j, $j = 1,\ldots,L$.

We carry out the approximation of (1) in a two step process. We begin in §2 with an exact treatment of the singularity, using a singularity division technique in an appropriate coordinate system. This gives us an explicit representation of the phase function required for MSP, and in §3 we describe the location of the critical points. In §4 we then proceed by applying a localized MSP approximation in the translated rotated coordinate system, which allows us to express (1) as an asymptotic series in ascending powers of $1/k$. We can also derive estimates for the remainder terms, and numerical results demonstrating the validity of these estimates appear in §5.

2 Singularity-free formulation

Under the assumption that the surface ∂D of the convex scatterer can be described globally in spherical coordinates, we write $\mathbf{x} \in \partial D$ as

$$\mathbf{x} = \mathbf{q}(\hat{\mathbf{x}}) = r(\theta, \phi)\mathbf{p}(\theta, \phi), \quad \theta \in [0, \pi], \quad \phi \in [0, 2\pi], \tag{6}$$

where $\hat{\mathbf{x}} \in \partial B$ (the unit sphere), is given by

$$\hat{\mathbf{x}} = \mathbf{p}(\theta, \phi) := (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)^{\mathrm{T}}, \quad \theta \in [0, \pi], \quad \phi \in [0, 2\pi].$$

With $\mathcal{J}$ being the Jacobian of $\mathbf{q}$, we get for any integrable ψ on ∂D,

$$\int_{\partial D} \psi(\mathbf{x})\,\mathrm{d}s(\mathbf{x}) = \int_{\partial B} \psi(\mathbf{q}(\hat{\mathbf{x}}))\mathcal{J}(\hat{\mathbf{x}})\,\mathrm{d}s(\hat{\mathbf{x}}), \tag{7}$$

and using (6) and (7), we rewrite (1) as

$$M\psi(\mathbf{q}(\hat{\mathbf{x}})) = \int_{\partial B} \frac{m(\mathbf{q}(\hat{\mathbf{x}}), \mathbf{q}(\hat{\mathbf{y}}))}{|\mathbf{q}(\hat{\mathbf{x}}) - \mathbf{q}(\hat{\mathbf{y}})|} \mathrm{e}^{\mathrm{i}k[|\mathbf{q}(\hat{\mathbf{x}}) - \mathbf{q}(\hat{\mathbf{y}})| + \hat{\mathbf{d}}.(\mathbf{q}(\hat{\mathbf{y}}) - \mathbf{q}(\hat{\mathbf{x}}))]} \psi(\mathbf{q}(\hat{\mathbf{y}}))\mathcal{J}(\hat{\mathbf{y}})\,\mathrm{d}s(\hat{\mathbf{y}}). \tag{8}$$

Recalling that $\hat{\mathbf{d}}$ is a fixed unit direction vector, $|\hat{\mathbf{d}}| = 1$, we write $\hat{\mathbf{d}} = \mathbf{p}(\theta_d, \phi_d)$, for some $\theta_d \in [0, \pi]$, $\phi_d \in [0, 2\pi]$. To simplify the weak singularity in (8), we then use the same transformation matrix as in [4]. For each $\hat{\mathbf{x}} \in \partial B$, we introduce the 3×3 orthogonal matrix $T_{\hat{\mathbf{x}}}$ which carries $\hat{\mathbf{x}}$ to the north pole: $T_{\hat{\mathbf{x}}}\hat{\mathbf{x}} = [0, 0, 1]^T =: \hat{\mathbf{n}}$. If $\hat{\mathbf{x}} = \mathbf{p}(\theta, \phi)$, an explicit form of $T_{\hat{\mathbf{x}}}$ is

$$T_{\hat{\mathbf{x}}} := \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{9}$$

Then writing $\hat{\mathbf{y}} = T_{\hat{\mathbf{x}}}^{-1}\hat{\mathbf{z}}$, we have $|\hat{\mathbf{x}} - \hat{\mathbf{y}}| = |\hat{\mathbf{x}} - T_{\hat{\mathbf{x}}}^{-1}\hat{\mathbf{z}}| = |T_{\hat{\mathbf{x}}}^{-1}(\hat{\mathbf{n}} - \hat{\mathbf{z}})| = |\hat{\mathbf{n}} - \hat{\mathbf{z}}|$. Since surface measure on ∂B is invariant under orthogonal transformations, we can rewrite (8) as

$$M\psi(\mathbf{q}(\hat{\mathbf{x}})) = \int_{\partial B} \frac{m(\mathbf{q}(\hat{\mathbf{x}}), \mathbf{q}(T_{\hat{\mathbf{x}}}^{-1}\hat{\mathbf{z}}))}{\tilde{f}_1(\hat{\mathbf{x}}, \hat{\mathbf{z}})} \mathrm{e}^{\mathrm{i}k[\tilde{f}_1(\hat{\mathbf{x}}, \hat{\mathbf{z}}) + \tilde{f}_2(\hat{\mathbf{x}}, \hat{\mathbf{d}}, \hat{\mathbf{z}})]} \psi(\mathbf{q}(T_{\hat{\mathbf{x}}}^{-1}\hat{\mathbf{z}}))\mathcal{J}(T_{\hat{\mathbf{x}}}^{-1}\hat{\mathbf{z}})\,\mathrm{d}s(\hat{\mathbf{z}}),$$

where $\tilde{f}_1(\hat{\mathbf{x}}, \hat{\mathbf{z}}) := |\mathbf{q}(\hat{\mathbf{x}}) - \mathbf{q}(T_{\hat{\mathbf{x}}}^{-1}\hat{\mathbf{z}})|$ and $\tilde{f}_2(\hat{\mathbf{x}}, \hat{\mathbf{d}}, \hat{\mathbf{z}}) := \hat{\mathbf{d}} \cdot (\mathbf{q}(T_{\hat{\mathbf{x}}}^{-1}\hat{\mathbf{z}}) - \mathbf{q}(\hat{\mathbf{x}}))$. We proceed by working out each term in the integrand using spherical coordinates.

With $\hat{\mathbf{z}} = \mathbf{p}(\theta', \phi')$ (and noting that θ' is then the angle between $\hat{\mathbf{n}}$ and $\hat{\mathbf{z}}$, equivalently the angle between $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$), and recalling (9) and that $T_{\hat{\mathbf{x}}}$ is an orthogonal transformation, it is straightforward to show that $T_{\hat{\mathbf{x}}}^{-1}\hat{\mathbf{z}} = \mathbf{p}(\theta'_x, \phi'_x)$, where θ'_x and ϕ'_x are functions of $\theta, \phi, \theta', \phi'$ satisfying

$$\sin\theta'_x \cos\phi'_x = \sin\theta'(\cos\theta\cos\phi\cos(\phi-\phi') + \sin\phi\sin(\phi-\phi')) + \cos\theta'\sin\theta\cos\phi,$$

$$\sin\theta'_x \sin\phi'_x = \sin\theta'(\cos\theta\sin\phi\cos(\phi-\phi') - \cos\phi\sin(\phi-\phi')) + \cos\theta'\sin\theta\sin\phi,$$

$$\cos\theta'_x = \cos\theta\cos\theta' - \sin\theta\sin\theta'\cos(\phi-\phi').$$

Using (6), we then get the following equalities;

$$\tilde{f}_1(\hat{\mathbf{x}}, \hat{\mathbf{z}}) = \sqrt{[r(\theta'_x, \phi'_x) - r(\theta, \phi)\cos\theta']^2 + [r(\theta, \phi)]^2 \sin^2\theta'} =: f_1(\theta, \phi, \theta', \phi'), \tag{10}$$

$$\begin{aligned} \tilde{f}_2(\hat{\mathbf{x}}, \hat{\mathbf{d}}, \hat{\mathbf{z}}) &= r(\theta'_x, \phi'_x)h(\theta_d, \phi_d, \theta'_x, \phi'_x) - r(\theta, \phi)h(\theta_d, \phi_d, \theta, \phi), \\ &=: f_2(\theta, \phi, \theta_d, \phi_d, \theta', \phi'). \end{aligned} \tag{11}$$

where $h(a, b, c, d) := \sin a \sin c \cos(b-d) + \cos a \cos c$. Hence using the notation

$$f(\theta, \phi, \theta_d, \phi_d, \theta', \phi') := f_1(\theta, \phi, \theta', \phi') + f_2(\theta, \phi, \theta_d, \phi_d, \theta', \phi'), \tag{12}$$

$$H(\theta, \phi, \theta', \phi') := m(r(\theta, \phi)\mathbf{p}(\theta, \phi), r(\theta'_x, \phi'_x)\mathbf{p}(\theta'_x, \phi'_x)) \frac{2\sin(\theta'/2)}{f_1(\theta, \phi, \theta', \phi')} \mathcal{J}(\mathbf{p}(\theta'_x, \phi'_x)), \tag{13}$$

and noting the 2π periodicity of the integrand with respect to ϕ', we get

$$\begin{aligned} &(M\psi)(r(\theta, \phi)\mathbf{p}(\theta, \phi)) = \\ &\int_{\phi}^{2\pi+\phi} \int_0^{\pi} H(\theta, \phi, \theta', \phi') \mathrm{e}^{\mathrm{i}kf(\theta, \phi, \theta_d, \phi_d, \theta', \phi')} \psi(r(\theta'_x, \phi'_x)\mathbf{p}(\theta'_x, \phi'_x)) \cos\frac{\theta'}{2}\, \mathrm{d}\theta'\, \mathrm{d}\phi'. \end{aligned} \tag{14}$$

Recalling the smoothness of $m(\cdot, \cdot)$ and $\mathcal{J}(\cdot)$, and noting that

$$\left|\frac{2\sin(\theta'/2)}{f_1(\theta, \phi, \theta', \phi')}\right| \leq \min\left(\frac{1}{|r(\theta, \phi)\cos(\theta'/2)|}, \frac{1}{|r(\theta'_x, \phi'_x) - r(\theta, \phi)\cos\theta'|}\right),$$

we have that $H(\theta, \phi, \theta', \phi')$ is a smooth (analytic) function in $\theta, \phi, \theta', \phi'$.

3 Evaluation of critical points

It is well known (see e.g. [6]) that the main contribution to the generalized Fourier integral (14) comes only from the values of the integrand at three types of *critical points*:

(i) Stationary points, at which $\nabla f := \left(\frac{\partial f}{\partial \theta'}, \frac{\partial f}{\partial \phi'}\right)^T = 0$.

(ii) Points on the boundary, at which one of the following equations holds.

$$\frac{\partial f(0,\phi')}{\partial \phi'} = 0; \quad \frac{\partial f(\pi,\phi')}{\partial \phi'} = 0; \quad \frac{\partial f(\theta',\phi)}{\partial \theta'} = 0; \quad \frac{\partial f(\theta',2\pi+\phi)}{\partial \theta'} = 0. \quad (15)$$

(iii) Corner points, namely $(0,\phi)$, $(0,2\pi+\phi)$, (π,ϕ), $(\pi,2\pi+\phi)$.

Here, we have written the phase f as a function of two (integration) variables θ', ϕ', using the fact that incident direction (θ_d, ϕ_d) and observed direction (θ, ϕ) are fixed. For closed surface scatterers, $f(0,\phi')$ and $f(\pi,\phi')$ are constant functions (see (10)–(12)), and hence the first two equations in (15) hold for all $\phi' \in [\phi, 2\pi+\phi]$. Since the phase function f is 2π periodic, to find remaining type (ii) critical points, we need to solve the scalar equation $\frac{\partial f(\theta',\phi)}{\partial \theta'} = 0$.

Next we consider critical points of type (i). For a fixed incident direction (θ_d, ϕ_d) and observed direction (θ, ϕ), using (10)–(12) and the notation

$$A := \sin\theta_d \cos\theta \cos(\phi-\phi_d) - \cos\theta_d \sin\theta, \quad B := \sin\theta_d \sin(\phi-\phi_d),$$

$$C(\theta',\phi') := \frac{r(\theta'_x,\phi'_x) - r(\theta,\phi)\cos\theta'}{r(\theta,\phi)\sin\theta'}, \; D(\phi') := A\cos(\phi-\phi') + B\sin(\phi-\phi'),$$

$$S(\theta',\phi') := \sin\theta \cos\theta' \cos(\phi-\phi') + \cos\theta \sin\theta', \quad T(\phi') := \sin\theta \sin(\phi-\phi'),$$

a little algebra (with each term evaluated at $a = \theta'_x$, $b = \phi'_x$) reveals that

$$\begin{aligned}\frac{\partial f}{\partial \theta'} =& \left[\frac{1}{[1+C(\theta',\phi')^2]^{1/2}} + \cos\theta' D(\phi') - \sin\theta' h(\theta,\phi,\theta_d,\phi_d)\right] r(\theta'_x,\phi'_x) \\ &+ \left[\frac{C(\theta',\phi')}{[1+C(\theta',\phi')^2]^{1/2}} + \sin\theta' D(\phi') + \cos\theta' h(\theta,\phi,\theta_d,\phi_d))\right] \\ &\times \left[\frac{S(\theta',\phi')}{\sqrt{[T(\phi')]^2+[S(\theta',\phi')]^2}}\frac{\partial}{\partial a} r(a,b) - \frac{T(\phi')}{[T(\phi')]^2+[S(\theta',\phi')]^2}\frac{\partial}{\partial b} r(a,b)\right], \quad (16)\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial \phi'} =& \sin\theta' \left\{(A\sin(\phi-\phi') - B\cos(\phi-\phi'))r(\theta'_x,\phi'_x)\right. \\ &+ \left[\frac{C(\theta',\phi')}{[1+C(\theta',\phi')^2]^{1/2}} + \sin\theta' D(\phi') + \cos\theta' h(\theta,\phi,\theta_d,\phi_d))\right] \\ &\times \left.\left[\frac{T(\phi')}{\sqrt{[T(\phi')]^2+[S(\theta',\phi')]^2}}\frac{\partial}{\partial a} r(a,b) + \frac{S(\theta',\phi')}{[T(\phi')]^2+[S(\theta',\phi')]^2}\frac{\partial}{\partial b} r(a,b)\right]\right\} \quad (17)\end{aligned}$$

In general, the nonlinear system (16)–(17) cannot be solved analytically. For notational simplicity and analytical calculations, in the remainder of this paper we will assume $r \equiv 1$ and $\hat{\mathbf{d}} = [0,0,1]^T$. Using (11)–(13), we get

$$H(\theta,\phi,\theta',\phi') = H(k,\theta') := \frac{1}{4\pi}\left[-\frac{1}{2} + \mathrm{i}\left(k\sin\frac{\theta'}{2} - 1\right)\right], \tag{18}$$

$$f(\theta,\phi,\theta',\phi') = 2\sin\frac{\theta'}{2} + \cos\theta(\cos\theta' - 1) - \sin\theta\sin\theta'\cos(\phi-\phi'), \tag{19}$$

$$\frac{\partial f}{\partial\theta'} = \cos\frac{\theta'}{2} - \cos\theta\sin\theta' - \sin\theta\cos\theta'\cos(\phi-\phi'), \tag{20}$$

$$\frac{\partial f}{\partial\phi'} = -\sin\theta\sin\theta'\sin(\phi-\phi') \tag{21}$$

In the following theorem, we describe critical points of type (i).

Theorem 3.1 *The stationary points $(\theta',\phi') \in [0,\pi]\times[\phi, 2\pi+\phi)$ of the phase function in (19) are as follows:*

- *If $\theta = 0$ then $\nabla f = 0$ for $(\theta',\phi') = (\pi/3,\phi'), (\pi,\phi')$.*
- *If $\theta \in (0,\pi/2)$ then there are five solutions of $\nabla f = 0$, given by $(\theta',\phi') = (\pi-2\theta,\phi)$, $((\pi-2\theta)/3,\phi)$, $((\pi+2\theta)/3,\phi+\pi)$, $(\pi,\phi+\pi/2)$ and $(\pi,\phi+3\pi/2)$.*
- *If $\theta = \pi/2$ then there are four solutions of $\nabla f = 0$, given by $(\theta',\phi') = (0,\phi)$, $(2\pi/3,\phi+\pi)$, $(\pi,\phi+\pi/2)$ and $(\pi,\phi+3\pi/2)$.*
- *If $\theta \in (\pi/2,\pi)$ then there are three solutions of $\nabla f = 0$, given by $(\theta',\phi') = ((\pi+2\theta)/3,\phi+\pi)$, $(\pi,\phi+\pi/2)$ and $(\pi,\phi+3\pi/2)$.*
- *If $\theta = \pi$ then $\nabla f = 0$ for $(\theta',\phi') = (\pi,\phi')$.*

Proof: First, suppose $\theta = \pi$. Then $\partial f/\partial\phi' = 0$ for all θ',ϕ', and $\frac{\partial f}{\partial\theta'} = \cos\frac{\theta'}{2}\left(1 + 2\sin\frac{\theta'}{2}\right) = 0$ if and only if $\theta' = \pi$. Next, suppose $\theta = 0$. Then again $\partial f/\partial\phi' = 0$ for all θ',ϕ', and $\frac{\partial f}{\partial\theta'} = \cos\frac{\theta'}{2}\left(1 - 2\sin\frac{\theta'}{2}\right) = 0$ for $\theta' = \pi$ or $\theta' = \pi/3$. Now, suppose $\theta \in (0,\pi)$. Then for $\partial f/\partial\phi' = 0$ to be satisfied, we must either have $\theta' = 0$, $\theta' = \pi$, or $\sin(\phi-\phi') = 0$. First, suppose $\theta' = 0$. Then $\partial f/\partial\phi' = 0$ for all ϕ', and $\partial f/\partial\theta' = 1 - \sin\theta\cos(\phi-\phi') = 0$ if and only if $\theta = \pi/2$ and $\cos(\phi-\phi') = 1$, which is satisfied only for $\phi' = \phi$. Next, suppose $\theta' = \pi$. Then again $\partial f/\partial\phi' = 0$ for all ϕ', and $\partial f/\partial\theta' = \sin\theta\cos(\phi-\phi') = 0$ if and only if $\cos(\phi-\phi') = 0$, i.e. if and only if $\phi' = \phi+\pi/2$ or $\phi' = \phi+3\pi/2$. Finally, suppose $\theta' \in (0,\pi)$. Then for $\partial f/\partial\phi' = 0$ to be satisfied we must have $\sin(\phi-\phi') = 0$, and hence $\phi' = \phi$ or $\phi' = \phi+\pi$. Supposing first that $\phi' = \phi$, we have $\frac{\partial f}{\partial\theta'} = \sin\left(\frac{\theta'+\pi}{2}\right) - \sin(\theta+\theta') = 0$ either if

$$\frac{\theta'+\pi}{2} = \theta + \theta' + 2n\pi, \quad n = 0,\pm1,\pm2,\ldots, \tag{22}$$

i.e. if $\theta' = \pi(1-4n) - 2\theta$, $n = 0,\pm1,\pm2,\ldots$, or if

$$\frac{\theta'+\pi}{2} = \pi - (\theta+\theta') + 2n\pi, \quad n = 0,\pm1,\pm2,\ldots, \tag{23}$$

i.e. if $\theta' = (\pi(1+4n) - 2\theta)/3$, $n = 0, \pm 1, \pm 2, \ldots$. For $n = 0$ we then have $\theta' = \pi - 2\theta$ or $\theta' = (\pi - 2\theta)/3$, each of which satisfies $\theta' \in [0, \pi]$ if and only if $\theta \in [0, \pi/2]$. For $n \neq 0$, all solutions θ' of (22) or (23) lie outside $[0, \pi]$. Finally we suppose that $\phi' = \phi + \pi$. Then $\frac{\partial f}{\partial \theta'} = \sin\left(\frac{\theta'+\pi}{2}\right) - \sin(\theta' - \theta) = 0$ either if

$$\frac{\theta' + \pi}{2} = \theta' - \theta + 2n\pi, \quad n = 0, \pm 1, \pm 2, \ldots, \tag{24}$$

i.e. if $\theta' = \pi(1 - 4n) + 2\theta$, $n = 0, \pm 1, \pm 2, \ldots$, or if

$$\frac{\theta' + \pi}{2} = \pi - (\theta' - \theta) + 2n\pi, \quad n = 0, \pm 1, \pm 2, \ldots, \tag{25}$$

i.e. if $\theta' = (\pi(1+4n) + 2\theta)/3$, $n = 0, \pm 1, \pm 2, \ldots$. For $n = 0$ we then have $\theta' = \pi + 2\theta$, which always lies outside $[0, \pi]$, or $\theta' = (\pi + 2\theta)/3$, which satisfies $\theta' \in [0, \pi]$. For $n \neq 0$, all solutions θ' of (24) or (25) lie outside $[0, \pi]$. □

4 Localized method of stationary phase and error analysis

Assuming for simplicity that $\theta \in (0, \pi/2) \cup (\pi/2, \pi)$, it follows from Theorem 3.1 that there are three stationary points, at

$$(\theta_s^1, \phi_s^1) := \left(\frac{\pi + 2\theta}{3}, \phi + \pi\right), \; (\theta_s^2, \phi_s^2) := \left(\pi, \phi + \frac{\pi}{2}\right), \; (\theta_s^3, \phi_s^3) := \left(\pi, \phi + \frac{3\pi}{2}\right),$$

and if $\theta \in (0, \pi/2)$ then there are two more stationary points at

$$(\theta_s^4, \phi_s^4) := (\pi - 2\theta, \phi), \quad (\theta_s^5, \phi_s^5) := \left(\frac{\pi - 2\theta}{3}, \phi\right).$$

Next we isolate these stationary points using a *partition of unity*. Taking pairwise disjoint neighborhoods Ω'_j of (θ_s^j, ϕ_s^j), $j = 1, \ldots, N(\theta)$, where

$$N(\theta) = \begin{cases} 3 \text{ if } & \pi/2 < \theta < \pi, \\ 5 \text{ if } & 0 < \theta < \pi/2, \end{cases} \tag{26}$$

and letting Ω_j be a small neighborhood of (θ_s^j, ϕ_s^j) such that $\overline{\Omega}_j \subset \Omega'_j$, we can construct a C^∞ neutralizing function χ_j (see [6, Ch.V, ex.7]) such that $\chi_j \equiv 1$ on Ω_j, $\chi_j \equiv 0$ outside Ω'_j. We then rewrite (14) as

$$(M\psi)(\mathbf{p}(\theta, \phi)) = \sum_{j=1}^{N(\theta)+1} (M_j\psi)(\mathbf{p}(\theta, \phi)), \tag{27}$$

where

$$(M_j\psi)(\mathbf{p}(\theta,\phi)) := \begin{cases} \int_\phi^{2\pi+\phi}\int_0^\pi G_j(\theta,\phi,\theta',\phi')\mathrm{e}^{\mathrm{i}kf(\theta,\phi,\theta',\phi')}\,\mathrm{d}\theta'\,\mathrm{d}\phi',\ j=1,\dots,N(\theta),\\ \int_\phi^{2\pi+\phi}\int_0^\pi g(\theta,\phi,\theta',\phi')\mathrm{e}^{\mathrm{i}kf(\theta,\phi,\theta',\phi')}\,\mathrm{d}\theta'\,\mathrm{d}\phi',\quad j=N(\theta)+1,\end{cases}$$

$$G_j(\theta,\phi,\theta',\phi') := \chi_j(\theta,\phi,\theta',\phi')H(k,\theta')\psi(\mathbf{p}(\theta'_x,\phi'_x))\cos(\theta'/2),\ j=1,\dots,N(\theta),$$

$$g(\theta,\phi,\theta',\phi') := \left[1-\sum_{j=1}^{N}\chi_j(\theta,\phi,\theta',\phi')\right]H(k,\theta')\psi(\mathbf{p}(\theta'_x,\phi'_x))\cos(\theta'/2).$$

Thus for $j=1,\dots,N(\theta)$ the domain of integration of M_j is a small region Ω_j, and the integrand of $M_{N(\theta)+1}$ is a C^∞ function with no stationary points.

We approximate $(M_j\psi)(\mathbf{p}(\theta,\phi))$, $j=1,\dots,N(\theta)$ using asymptotic expansions about each stationary point. First we define

$$(\widehat{M}_1\psi)(\mathbf{p}(\theta,\phi)) := -\frac{2\pi\mathrm{i}G_1(\theta,\phi,\theta_s^1,\phi_s^1)\mathrm{e}^{\mathrm{i}kf(\theta,\phi,\theta_s^1,\phi_s^1)}}{k\left[k\frac{3}{2}\cos\left(\frac{\theta-\pi}{3}\right)\sin\theta\sin\left(\frac{\pi+2\theta}{3}\right)\right]^{1/2}}, \tag{28}$$

$$(\widehat{M}_2\psi)(\mathbf{p}(\theta,\phi)) := 0, \qquad (\widehat{M}_3\psi)(\mathbf{p}(\theta,\phi)) := 0, \tag{29}$$

$$(\widehat{M}_4\psi)(\mathbf{p}(\theta,\phi)) := \frac{2\pi\mathrm{i}G_4(\theta,\phi,\theta_s^4,\phi_s^4)\mathrm{e}^{\mathrm{i}kf(\theta,\phi,\theta_s^4,\phi_s^4)}}{k\left[\frac{1}{2}\cos\theta\sin\theta\sin2\theta\right]^{1/2}}, \tag{30}$$

$$(\widehat{M}_5\psi)(\mathbf{p}(\theta,\phi)) := \frac{2\pi G_5(\theta,\phi,\theta_s^5,\phi_s^5)\mathrm{e}^{\mathrm{i}kf(\theta,\phi,\theta_s^5,\phi_s^5)}}{k\left[\frac{3}{2}\cos\left(\frac{\theta+\pi}{3}\right)\sin\theta\sin\left(\frac{\pi-2\theta}{3}\right)\right]^{1/2}}. \tag{31}$$

The following result gives the power of approximating $(M_j\psi)$ by $\widehat{M}_j\psi$, for $j=1,\dots,N(\theta)$.

Theorem 4.1 *For $j=1,\dots,N(\theta)$, and for large k,*

$$(M_j\psi)(\mathbf{p}(\theta,\phi)) - (\widehat{M}_j\psi)(\mathbf{p}(\theta,\phi) = O\left(\frac{1}{k^2}\right).$$

Proof: First we note that, with $f := f(\theta,\phi,\theta',\phi')$,

$$\frac{\partial^2 f}{\partial\theta'^2}\frac{\partial^2 f}{\partial\phi'^2} - \frac{\partial^2 f}{\partial\theta'\partial\phi'}\begin{cases} >0 \text{ if } (\theta',\phi')=(\theta_s^i,\phi_s^i)\ \ i=1,4\\ <0 \text{ if } (\theta',\phi')=(\theta_s^j,\phi_s^j)\ \ j=2,3,5\end{cases}$$

Thus (θ_s^1,ϕ_s^1) is a local maximum, (θ_s^4,ϕ_s^4) is a local minimum, and (θ_s^j,ϕ_s^j), $j=2,3,5$, are each saddle points. Following [6, Chapter VIII] we can write a series expansion for M_j, $j=1,\dots,N$, in increasing powers of $1/k$, with in each case the leading order term given by $\widehat{M}_j$. Noting that $\cos\theta_s^2/2=\cos\theta_s^3/2=0$, the result follows. □

To evaluate $(M_{N+1}\psi)(\mathbf{p}(\theta,\phi))$, we take advantage of the fact that ∇f is bounded away from zero on the range of integration to write the integral as a series of line integrals, increasing in powers of $1/k$, and define

$$(\widehat{M}_{N(\theta)+1}\psi)(\mathbf{p}(\theta,\phi)) =$$
$$\frac{2\pi \mathrm{i} H(k,0)\psi(\mathbf{p}(\theta,\phi))}{k|\cos\theta|} - \frac{2\pi H(k,0)\psi(\mathbf{p}(\theta,\phi))}{k^2\cos^4\theta}\left[1+\frac{1}{2}\sin^2\theta\right] - \frac{2\pi\frac{\partial H}{\partial\theta'}(k,0)\psi(\mathbf{p}(\theta,\phi))}{k^2|\cos^3\theta|}$$
$$-\frac{H(k,0)}{k^2}\int_{\phi}^{2\pi+\phi}\frac{\partial\psi(\mathbf{p}(\theta'_x,\phi'_x))}{\partial\theta'}\bigg|_{\theta'=0}\frac{\mathrm{d}\phi'}{(1-\sin\theta\cos(\phi-\phi'))^2}. \tag{32}$$

To analyze the error in approximation of $(M_{N(\theta)+1}\psi)(\mathbf{p}(\theta,\phi))$ by $(\widehat{M}_{N(\theta)+1}\psi)(\mathbf{p}(\theta,\phi))$, we require the following result.

Lemma 4.2 *For any constant c,*

$$K_m(\theta) := \int_c^{2\pi+c}\frac{1}{(1-\sin\theta\cos(y-c))^m}\,\mathrm{d}y =$$
$$\frac{2\pi}{|\cos\theta|^{2m-1}}\sum_{j=0}^{m-1}\frac{(2j-1)!!}{j!}\begin{pmatrix} m-1 \\ j \end{pmatrix}\sin^j\theta(1-\sin\theta)^{m-1-j}, \quad m=1,2,\cdots,$$

where $(2j-1)!!=1$ *if* $j=0$, *and* $(2j-1)!! := 1.3.5\ldots(2j-3)(2j-1), \quad j\geq 1$.

Proof: Making the substitution $t=\tan((y-c)/2)$, and defining $a^2 := \frac{1-\sin\theta}{1+\sin\theta}$,

$$K_m(\theta) = \frac{4}{(1+\sin\theta)^m}\sum_{j=0}^{m-1}\begin{pmatrix} m-1 \\ j \end{pmatrix}(1-a^2)^j\int_0^{\infty}\frac{1}{(a^2+t^2)^{j+1}}\,\mathrm{d}t.$$

Noting that $\int_0^\infty \frac{1}{(a^2+t^2)^{j+1}}\,\mathrm{d}t = \frac{1}{a^{2j+1}}\frac{(2j-1)!!\pi}{2^{j+1}j!}$, the result follows. □

We are now ready to prove the main result of this section.

Theorem 4.3 *For fixed $\theta\in(0,\pi/2)\cup(\pi/2,\pi)$ and $\phi\in[0,2\pi]$, there exist constants $C_1>0$, $C_2(\theta,\phi)>0$, each bounded independently of k, such that for k sufficiently large*

$$\left|(M_{N+1}\psi)(\mathbf{p}(\theta,\phi)) - (\widehat{M}_{N(\theta)+1}\psi)(\mathbf{p}(\theta,\phi))\right| \leq \frac{1}{k}\left(\frac{C_1}{\sin^2\theta}+\frac{C_2(\theta,\phi)}{k}\right). \tag{33}$$

Proof: Since $\nabla f\neq 0$ for $(\theta',\phi')\in\mathrm{supp}(g)$, it follows from the divergence theorem and the identity $\nabla.(\mathbf{u}\mathrm{e}^{\mathrm{i}kf}) = (\nabla.\mathbf{u})\mathrm{e}^{\mathrm{i}kf} + \mathrm{i}kg\mathrm{e}^{\mathrm{i}kf}$, where $\mathbf{u}=\mathbf{u}_0 := \frac{\nabla f}{|\nabla f|^2}g$, $g := g(\theta,\phi,\theta',\phi')$, that for $n=1,2,\ldots$,

$$(M_{N+1}\psi)(\mathbf{p}(\theta,\phi)) = -J(n) + \left(\frac{\mathrm{i}}{k}\right)^n\iint_{\mathrm{supp}(g)} g_n\mathrm{e}^{\mathrm{i}kf}\,\mathrm{d}\theta'\,\mathrm{d}\phi', \tag{34}$$

where

$$J(n) := \sum_{s=0}^{n-1}\left(\frac{\mathrm{i}}{k}\right)^{s+1}\int_{\Gamma}(\mathbf{u}_s.\mathbf{n})\mathrm{e}^{\mathrm{i}kf}\,\mathrm{d}\sigma, \quad g_{s+1} := (\nabla.\mathbf{u}_s), \quad \mathbf{u}_{s+1} := \frac{\nabla f}{|\nabla f|^2}g_{s+1}, \tag{35}$$

Γ is the positively oriented (anticlockwise) boundary of $\mathrm{supp}(g)$, σ is the arc length of Γ, and $\mathbf{n} := (n_1, n_2)$ is the unit outward normal vector to Γ. We immediately deduce that for $n = 1, 2, \ldots$,

$$|(M_{N+1}\psi)(\mathbf{p}(\theta,\phi)) + J(n)| \leq \frac{1}{k^n}\left|\int\int_{\mathrm{supp}(g)} g_n \mathrm{e}^{\mathrm{i}kf}\,\mathrm{d}\theta'\,\mathrm{d}\phi'\right| \leq \frac{C(\theta,\phi)}{k^{n+1}}\|g_n\|_\infty. \tag{36}$$

Next we evaluate

$$\int_{\Gamma}(\mathbf{u}_s.\mathbf{n})\mathrm{e}^{\mathrm{i}kf}\,\mathrm{d}\sigma = \int_{\Gamma}\frac{n_1 f_{\theta'} + n_2 f_{\phi'}}{f_{\theta'}^2 + f_{\phi'}^2}\mathrm{e}^{\mathrm{i}kf}g_s\,\mathrm{d}\sigma, \quad \text{for } s = 0, 1. \tag{37}$$

As shown in Figure 1 (for $\theta \in (\pi/2, \pi)$), $\mathrm{supp}(g)$ is bounded by the lines $\phi' = \phi$, $\phi' = 2\pi + \phi$, $\theta' = 0$, $\theta' = \pi$ and the supports of $1 - \chi_j(\theta, \phi, \theta', \phi')$, $j = 1, \ldots, N$.

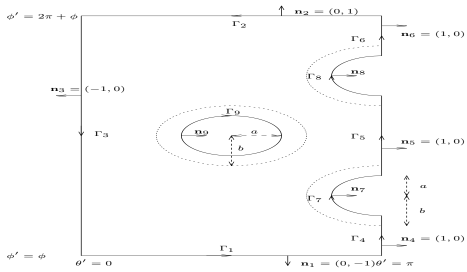

Figure 1: Domain of integration supp(g), for $\theta \in (\pi/2, \pi)$.

The contributions to (37) from the sections of Γ corresponding to $\phi' = \phi$ and $\phi' = 2\pi + \phi$ (Γ_1 and Γ_2 in Figure 1) are both zero, since for $\phi' = \phi$ and $\phi' = 2\pi + \phi$ we have $n_1 = 0$ and $f_{\phi'} = 0$. On the sections of Γ corresponding to $\theta' = 0$ and $\theta' = \pi$ (Γ_3, Γ_4, Γ_5 and Γ_6 in Figure 1) we have

$$\frac{n_1 f_{\theta'} + n_2 f_{\phi'}}{f_{\theta'}^2 + f_{\phi'}^2}\mathrm{e}^{\mathrm{i}kf} = \begin{cases} -1/(1 - \sin\theta\cos(\phi - \phi')), & \text{on } \theta' = 0, \\ \mathrm{e}^{\mathrm{i}k(2 - 2\cos\theta)}/\sin\theta\cos(\phi - \phi'), & \text{on } \theta' = \pi. \end{cases} \tag{38}$$

Recalling (35),

$$g_{s+1} = P(\theta', \phi')g_s + \frac{f_{\theta'}}{f_{\theta'}^2 + f_{\phi'}^2}\frac{\partial g_s}{\partial \theta'} + \frac{f_{\phi'}}{f_{\theta'}^2 + f_{\phi'}^2}\frac{\partial g_s}{\partial \phi'}, \tag{39}$$

where

$$P(\theta',\phi') := \frac{f_{\theta'\theta'} + f_{\phi'\phi'}}{f_{\theta'}^2 + f_{\phi'}^2} - 2\frac{f_{\theta'}^2 f_{\theta'\theta'} + 2f_{\theta'}f_{\phi'} + f_{\phi'}^2 f_{\phi'\phi'}}{(f_{\theta'}^2 + f_{\phi'}^2)^2}.$$

From the definition of χ_j, $j = 1,\ldots,N$, g and all its derivatives, and hence g_s, $s = 0,1,\ldots$, then vanish on all other sections of Γ (Γ_7, Γ_8 and Γ_9 in Figure 1, plus four other semicircles in the case $\theta \in (0,\pi/2)$). Thus

$$\int_\Gamma (\mathbf{u}_s.\mathbf{n})\mathrm{e}^{\mathrm{i}kf}\,\mathrm{d}\sigma = \int_\phi^{2\pi+\phi} \frac{g_s(\theta,\phi,0,\phi')}{1-\sin\theta\cos(\phi-\phi')}\,\mathrm{d}\phi' + \frac{\mathrm{e}^{\mathrm{i}k(2-2\cos\theta)}}{\sin\theta}\int_\phi^{2\pi+\phi} \frac{g_s(\theta,\phi,\pi,\phi')}{\cos(\phi-\phi')}\,\mathrm{d}\phi'. \quad (40)$$

Using (39),

$$g_{s+1}(\theta,\phi,0,\phi') = \frac{\cos\theta}{(1-\sin\theta\cos(\phi-\phi'))^2} g_s(\theta,\phi,0,\phi') + \frac{\frac{\partial g_s}{\partial\theta'}(\theta,\phi,0,\phi')}{1-\sin\theta\cos(\phi-\phi')}, \quad (41)$$

$$g_{s+1}(\theta,\phi,\pi,\phi') = \left(\frac{1/2-\cos\theta}{\sin^2\theta\cos^2(\phi-\phi')}\right) g_s(\theta,\phi,\pi,\phi') + \frac{\frac{\partial g_s}{\partial\theta'}(\theta,\phi,\pi,\phi')}{\sin\theta\cos(\phi-\phi')}, \quad (42)$$

and since $\mathbf{p}(\theta'_x,\phi'_x)|_{\theta'=0} = \mathbf{p}(\theta,\phi)$ and $\mathbf{p}(\theta'_x,\phi'_x)|_{\theta'=\pi} = \mathbf{p}(\pi-\theta,\phi)$,

$$g(\theta,\phi,0,\phi') = H(k,0)\psi(\mathbf{p}(\theta,\phi)), \qquad g(\theta,\phi,\pi,\phi') = 0,$$

$$\begin{aligned} g_1(\theta,\phi,0,\phi') = & \left[\frac{H(k,0)\cos\theta}{(1-\sin\theta\cos(\phi-\phi'))^2} + \frac{H'(k,0)}{(1-\sin\theta\cos(\phi-\phi'))}\right]\psi(\mathbf{p}(\theta,\phi)) \\ & + \left[\frac{H(k,0)}{(1-\sin\theta\cos(\phi-\phi'))}\right] \left.\frac{\partial\psi(\mathbf{p}(\theta'_x,\phi'_x))}{\partial\theta'}\right|_{\theta'=0}, \\ g_1(\theta,\phi,\pi,\phi') = & \frac{-\left[1-\sum_{j=1}^N \chi_j(\theta,\phi,\theta',\phi')\right] H(k,\pi)\psi(\mathbf{p}(\pi-\theta,\phi)}{2\sin\theta\cos(\phi-\phi')}. \end{aligned}$$

Using these in (40), for $s = 0,1$,

$$\int_\Gamma (\mathbf{u}_0.\mathbf{n})\mathrm{e}^{ikf}\,\mathrm{d}\sigma = H(k,0)\psi(\mathbf{p}(\theta,\phi)) \int_\phi^{2\pi+\phi} \frac{1}{(1-\sin\theta\cos(\phi-\phi'))}\,\mathrm{d}\phi', \quad (43)$$

$$\begin{aligned} & \int_\Gamma (\mathbf{u}_1.\mathbf{n})\mathrm{e}^{ikf}\,\mathrm{d}\sigma = \\ & \int_\phi^{2\pi+\phi} \frac{H(k,0)\psi(\mathbf{p}(\theta,\phi))\cos\theta}{(1-\sin\theta\cos(\phi-\phi'))^3}\,\mathrm{d}\phi' + \int_\phi^{2\pi+\phi} \frac{H'(0)\psi(\mathbf{p}(\theta,\phi))}{(1-\sin\theta\cos(\phi-\phi'))^2}\,\mathrm{d}\phi' \\ & + \int_\phi^{2\pi+\phi} \left.\frac{\partial\psi(\mathbf{p}(\theta'_x,\phi'_x))}{\partial\theta'}\right|_{\theta'=0} \frac{H(k,0)}{(1-\sin\theta\cos(\phi-\phi'))^2}\,\mathrm{d}\phi' \end{aligned}$$

$$+\frac{\mathrm{e}^{\mathrm{i}k(2-2\cos\theta)}H(k,\pi)\psi(\mathbf{p}(\pi-\theta,\phi))}{2\sin^2\theta}\int_{\phi}^{2\pi+\phi}\frac{\left[1-\sum_{j=1}^{N}\chi_j(\theta,\phi,\theta',\phi')\right]}{\cos^2(\phi-\phi')}\,\mathrm{d}\phi'. \qquad (44)$$

Applying Lemma 4.2 and the fact that $H(k,\pi)$ is of order k in (43) and (44), the result (33) follows from (35) (with $n=2$), (34) and (32). $\square$

Remark 4.4 *Using (40), (41) and Lemma 4.2, we see that for* $\theta=\pi/2\pm\delta$, *the leading order term of* $\int_\Gamma(\mathbf{u}_s.\mathbf{n})\mathrm{e}^{\mathrm{i}kf}\,\mathrm{d}\sigma$ *for fixed* k *as* $\delta\to 0$ *is of order*

$$\int_{\phi}^{2\pi+\phi}\frac{\cos^s\theta}{(1-\sin\theta\cos(\phi-\phi'))^{2s+1}}\,\mathrm{d}\phi'\sim\frac{\cos^s\theta}{|\cos\theta|^{4s+1}}\sim\frac{1}{\delta^{3s+1}}.$$

Thus as $\delta\to 0$, *with* k *fixed, each term in* $J(n)$ *(see (35)) is of order* $\frac{1}{k^{s+1}\delta^{3s+1}}=\frac{\delta^2}{(k\delta^3)^{s+1}}$. *Hence for fixed* k, *we require* $\delta\geq Ck^{-1/3}$, *for some constant* $C>0$.

5 Numerical results

We demonstrate our approach by computing efficient approximations to highly oscillatory weakly singular integrals $M\psi(\mathbf{x})$ in (1), with m given by the acoustic scattering kernel (5), for a spherical scatterer of radius 1 at 1000 observed directions $\mathbf{x}=\mathbf{p}(\theta,0)$, for various wavenumbers $k>10000$.

Analytical solutions of these integrals are not known even for the constant density $\psi\equiv 1$. However, in the rotated coordinate system the outer part of these surface integrals (see (14)) can, for the case of a sphere and $\psi\equiv 1$, be evaluated exactly using the Bessel functions of order zero: $\int_0^{2\pi}\mathrm{e}^{\mathrm{i}ka\cos y}\,\mathrm{d}y=2\pi J_0(ka)$. For comparison purposes, we evaluated the inner part of the surface integrals to a very high accuracy, using the Gaussian quadrature with 30 nodes per half wavelength. In this section, we take the resulting computed number to be the exact value of $(M\psi)(\mathbf{p}(\theta,0))$, with $\psi\equiv 1$.

Using (27), Theorems 4.1 and 4.3, our approximation $(M_{app}\psi)(\mathbf{p}(\theta,0))$ to $(M\psi)(\mathbf{p}(\theta,0))$ for $\theta\in(0,\pi)$, $\theta\neq\pi/2$, is defined by

$$(M_{app}\psi)(\mathbf{p}(\theta,0)):=\sum_{j=1}^{N(\theta)+1}(\widehat{M_j}\psi)(\mathbf{p}(\theta,0)), \qquad (45)$$

where $\widehat{M_j}\psi$ for $j=1,\cdots N(\theta)+1$ are given by (28)–(32), and $N(\theta)$ is as defined in (26). ¿From Theorems 4.1 and 4.3 and recalling Remark 4.4, we would expect that for $|\theta-\pi/2|>Ck^{-1/3}$ for some fixed constant $C>0$,

$$E(k,\theta):=\frac{|(M\psi)(\mathbf{p}(\theta,0))-(M_{app}\psi)(\mathbf{p}(\theta,0))|}{|(M\psi)(\mathbf{p}(\theta,0))|}\leq\frac{c(\theta)}{k}. \qquad (46)$$

We computed the exact value of $(M\psi)(\mathbf{p}(\theta,0))$ for $\theta \in S_{1001}$, where the set S_{1001} consists of 1001 equally spaced points on $[0,\pi]$, including the end points. We computed the approximation $(M_{app}\psi)(\mathbf{p}(\theta,0))$ with $\theta \in S_{1001} \setminus \{0,\pi/2,\pi\}$.

In Figures 2 and 3, we plot the exact value $(M\psi)(\mathbf{p}(\theta,0))$ for $\theta \in S_{1001}\setminus\{0,\pi\}$, and approximate solution $(M_{app}\psi)(\mathbf{p}(\theta,0))$ for $\theta \in S_{1001} \setminus \{0,\pi\}$, $|\theta-\pi/2| > 10k^{-1/3}$, for $k = 1,310,720$ and $k = 2,621,440$. Our approximations are seen to be qualitatively correct (in the sense that the crosses for the approximate values lie inside the circles representing the exact values) outside a region of width of the order of $k^{-1/3}$ around $\theta = \pi/2$. Evaluation of just the one dimensional inner integral for the exact solution of $(M\psi)(\mathbf{p}(\theta,0))$ with $\theta \in S_{1001}$, $k = 1,310,720$ and $k = 2,621,440$ took over **44 hours** and **94 hours** of CPU time respectively on a AMD Opteron 2.0Ghz computer, while our approximation $(M_{app}\psi)(\mathbf{p}(\theta,0))$ with $\theta \in S_{1001} \setminus \{0,\pi/2,\pi\}$ was computed for both the cases in less than **0.03 seconds**.

In Figure 4 we plot the error $E(k,\theta)$ for $|\theta-\pi/2| > 10k^{-1/3}$, for $k = 10240$, $k = 40960$, $k = 163840$, $k = 655360$ and $k = 1,310,720$, to demonstrate efficiency of our formula for computing (1) within a few seconds of CPU time.

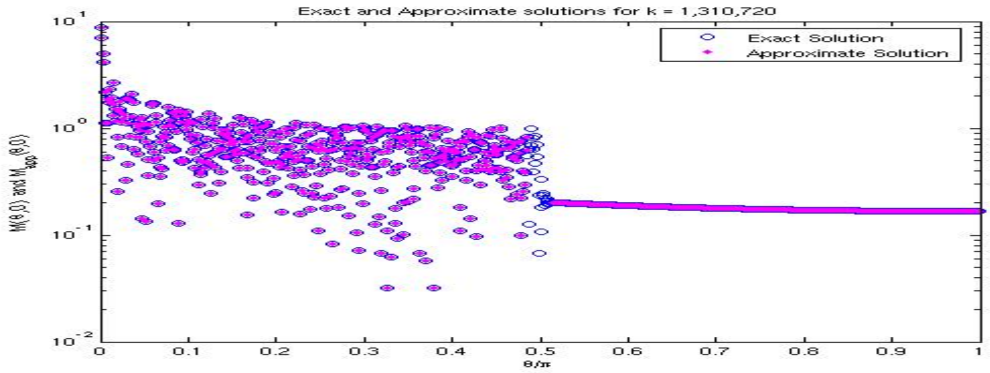

Figure 2: Exact and approximate solutions for $k = 1,310,720$.

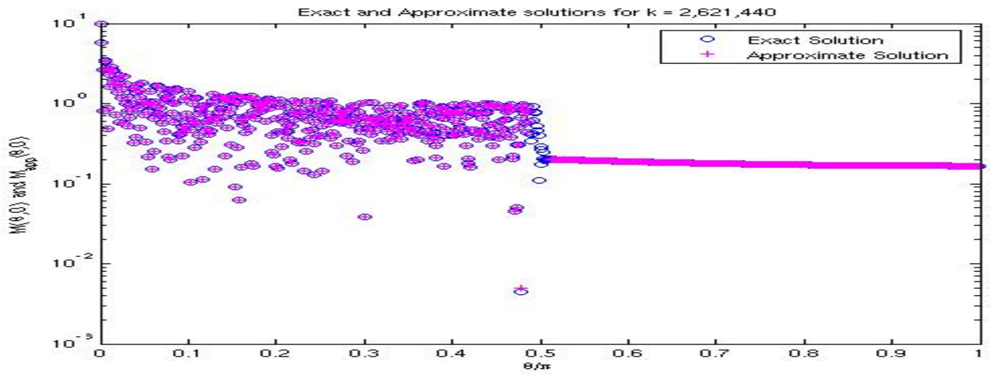

Figure 3: Exact and approximate solutions for $k = 2,621,440$.

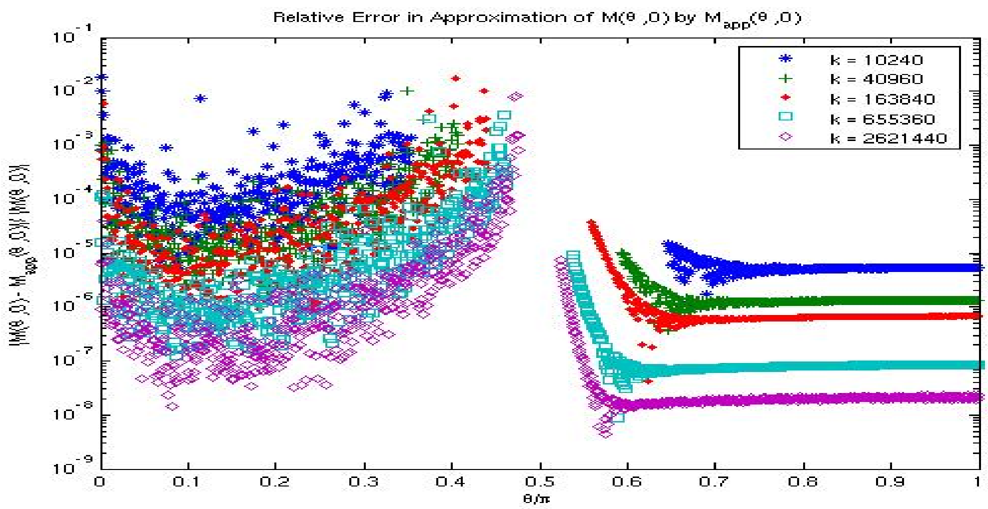

Figure 3: Errors $E(k,\theta)$ for $|\theta - \pi/2| > 10k^{-1/3}$, various $k > 10000$.

6 Conclusions

Outside of a band of width $Ck^{-1/3}$ around the shadow boundary $\theta = \pi/2$, our approximation scheme is very accurate. The computational time required to approximate the integral at a thousand points remains constant for all values of k, in stark contrast to the severe increase in computational cost needed to maintain accuracy as k increases for standard quadrature schemes. The application of our integration scheme to the solution of the full scattering problem (2) will be considered in a future work.

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